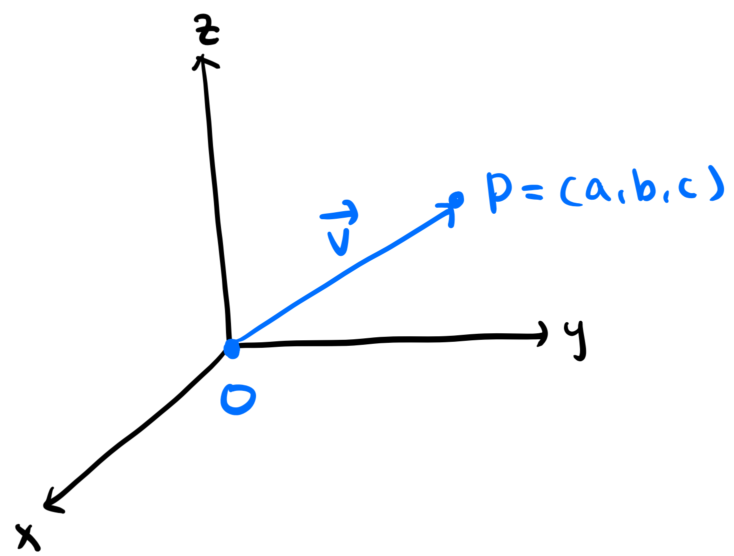
12.2. Vectors

Def (1) A vector is defined as

- an object with direction and length
- a point in a coordinate system



(2) The magnitude of $\vec{v}=(a,b,c)$ is

$$|\vec{v}| := \sqrt{a^2+b^2+c^2} = \text{length}$$

Pythagorean theorem

(3) A unit vector is a vector of magnitude 1.

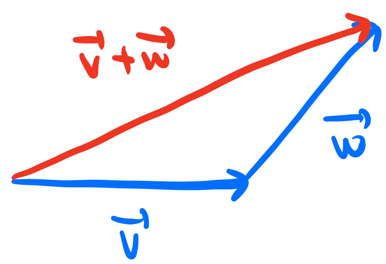
(4) The standard unit vectors in $\mathbb{R}^3$ are

$$\vec{i}=(1,0,0),\quad \vec{j}=(0,1,0),\quad \vec{k}=(0,0,1)$$

Def (Basic operations on vectors)

(1) The <u>sum</u> of $\vec{v} = (a_1, b_1, c_1)$ and $\vec{w} = (a_2, b_2, c_2)$ is

$$\vec{v} + \vec{w} := (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2)$$

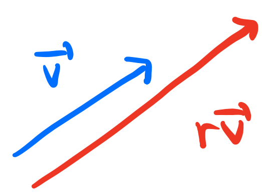


"head-to-tail combination"

(2) The <u>scalar multiple</u> of $\vec{v} = (a, b, c)$ by

a number r is

$$r\vec{v} := (ra, rb, rc)$$



"scaling by r"

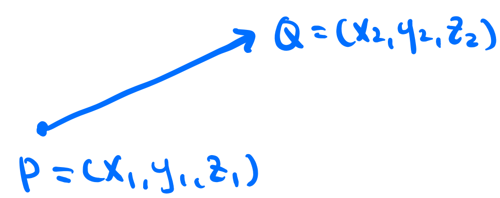
Note (1) $(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$.

(2) $r\vec{v}$ is parallel to $\vec{v}$ with length multiplied by $|r|$.

(3) $\vec{v}$ and $-\vec{v}$ have opposite directions.

★ **Prop** For $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$,

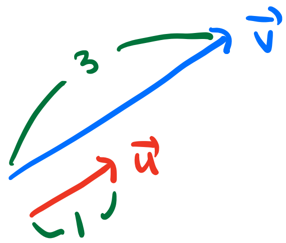
$$\overrightarrow{PQ} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$



Ex Consider the vector $\vec{v} = \vec{i} - 2\vec{j} + 2\vec{k}$

(1) Find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

Sol



$\vec{v} = (1, -2, 2)$

$|\vec{v}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3.$

To get the unit vector $\vec{u}$ of length 1,

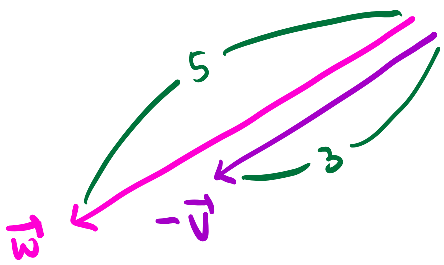
we should scale $\vec{v}$ by $\frac{1}{3}$ $\left(= \frac{1}{|\vec{v}|}\right)$

$\Rightarrow \vec{u} = \frac{1}{3}\vec{v} = \boxed{\frac{1}{3}(-1, 2, 2)}$

Rmk More generally, the unit vector of any vector $\vec{v}$ is given by $\boxed{\vec{u} = \frac{\vec{v}}{|\vec{v}|}}$

(2) Find the vector $\vec{w}$ of length 5 in the opposite direction of $\vec{v}$.

Sol



The vector $-\vec{v}$ is in the opposite direction of $\vec{v}$.

To get the vector $\vec{w}$ of length 5, we should scale $-\vec{v}$ by $\frac{5}{3}$.

$\Rightarrow \vec{w} = \frac{5}{3}(-\vec{v}) = -\frac{5}{3}\vec{v} = \boxed{-\frac{5}{3}(-1, 2, 2)}$

12.3. The dot product

Def The dot product (or scalar product) of

$\vec{v} = (a_1, b_1, c_1)$ and $\vec{w} = (a_2, b_2, c_2)$ is

$$\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2 \rightarrow \text{scalar}$$

Prop (Algebraic properties of the dot product)

(1) $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

(2) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(3) $\vec{u} \cdot (\underbrace{\vec{v} + \vec{w}}_{\text{vector addition}}) = \vec{u} \cdot \vec{v} \underbrace{+}_{\text{scalar addition}} \vec{u} \cdot \vec{w}$

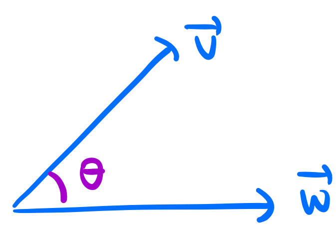
(4) $(r\vec{v}) \cdot \vec{w} = r(\vec{v} \cdot \vec{w})$ where r is a number.

(5) $\vec{v} \cdot \vec{0} = 0$ where $\vec{0} = (0, 0, 0)$

Thm Let θ be the angle between $\vec{v}$ and $\vec{w}$.

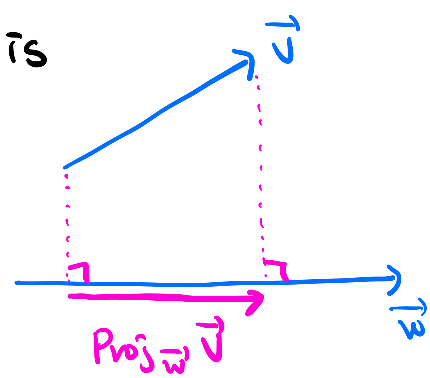
(1) $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$

(2) $\theta = \cos^{-1}\left(\dfrac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right)$



Prop The projection of $\vec{v}$ onto $\vec{w}$ is

$$\text{Proj}_{\vec{w}} \vec{v} = \underbrace{\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}}_{\text{scalar}} \vec{w}$$

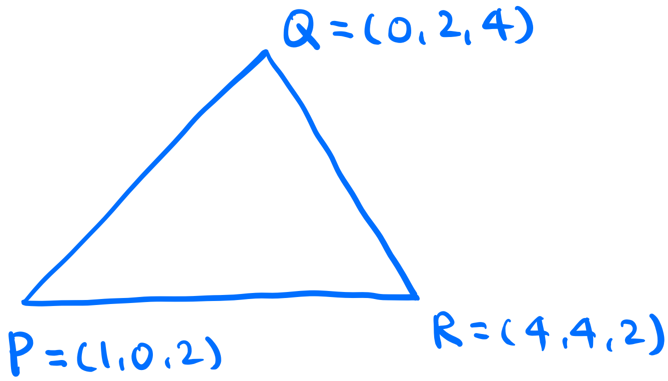


Ex Consider the triangle with vertices at

$P = (1, 0, 2)$, $Q = (0, 2, 4)$, $R = (4, 4, 2)$

Find the angle at P.

Sol



$$\angle P = \cos^{-1}\left(\frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|}\right)$$

$$\vec{PQ} = (-1, 2, 2), \quad \vec{PR} = (3, 4, 0)$$

$$\vec{PQ}\cdot\vec{PR} = (-1)\cdot 3 + 2\cdot 4 + 2\cdot 0 = 5$$

$$|\vec{PQ}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$$

$$|\vec{PR}| = \sqrt{3^2 + 4^2 + 0^2} = 5$$

$$\Rightarrow \angle P = \cos^{-1}\left(\frac{5}{3\cdot 5}\right) = \boxed{\cos^{-1}\left(\frac{1}{3}\right)}$$

Rmk In Math 215, all angles are measured in radians unless stated otherwise.